Ternary "Quaternions" and Ternary $TU(3)$ algebra

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Abstract

To construct ternary "quaternions" following Hamilton we must introduce two "imaginary "units, q_1 and q_2 with propeties $q_1^n = 1$ and $q_2^m = 1$. The general is enough difficult, and we consider the $m = n = 3$. This case gives us the example of non-Abelian groupas was in Hamiltonian quaternion. The Hamiltonian quaternions help us to discover the $SU(2) = S^3$ group and also the group $L(2, C)$. In ternary case we found the generalization of U(3) which we called $TU(3)$ group and also we found the the SL(3,C) group. On the matrix language we are going from binary Pauly matrices to three dimensional nine matrices which are called by nonions. This way was initiated by algebraic classification of CY_m-spaces for all m=3,4,...where in reflexive Newton polyhedra we found the Berger graphs which gave in the corresponding Cartan matrices the longest simple roots $B_{ii} = 3, 4, ..$ comparing with the case of binary algebras in which the Cartan diagonal element is equal 2, *i.e.* $A_{ii} = 2$.

We will discuss the following results

- $_n$ quaternization of R^n spaces
- Ternary "quaternion" structure structure and the invariant surfaces
- New geometry and non-Abelian N-ary algebras/symmetries
- Root system of a new ternary $TU(3)$ algebra
- N-ary Clifford algebras

Contents

1 Introduction

The complexification of R^n Euclidean spaces gave us the generalization of $U(1)$ group to the n-parameter Abelian groups $U_n = \exp(\alpha_1 q + \alpha_2 q^2 + ... + \alpha_{n-1} q^{n-1})$ [1, 2, 3] The Hamilton procedure is going to discover non-Abelian groups [4] . This question we will discuss in our article.

In all these approaches there were used a wide class of simple classical Lie algebras, whose Cartan-Killing classification contains four infinite series $A_r = sl(r+1)$,$B_r = so(2n+1)$, $C_r = sp(2r)$, $D_r = so(2r)$ and five exceptional algebras G_2, F_4, E_6, E_7, E_8. There were used some ways to study such classification. We can remind some of them, one way is through the theory of numbers and Clifford algebras, the second is the geometrical way, and at last, the third is through the theory of Cartan matrices and Dynkin diagramms.

Before to show a new root system for ternary non-Abelian algebra (in our example of $(TU(3)$)it is very useful to remind the theory of simple roots in binary Cartan Lie algebra. We well know how the simple roots allows us to reconstruct all root system and, consequently, all commutation relations in the corresponding CLA.

The finite-dimensional Lie algebra g of a compact simple Lie group G is determined by the following binary commutation relations

$$[T_a, T_b]_{Z_2} = \mathbf{i} f_{abc} T_c, \tag{1}$$

where the basis of generators $\{T_a\}$ of g is assumed to satisfy the orthonormality condition:

$$Tr(T_a T_b) = y\, \delta_{ab}. \tag{2}$$

The constant y depends on the representation chosen.

The standard way of choosing a basis for g is to define the maximal set h, $[hh] = 0$, of commuting Hermitiam generators, H_i, (i=1,2,...,r).

$$[H_i, H_j] = 0, \qquad 1 \le i,\, j \le r. \tag{3}$$

This set h of H_i forms the Abelian Cartan subalgebra (CSA). The dimension r is called the rank of g (or G). Then we can extend a basis taking complex generators $E_{\vec{\alpha}}$, such that

$$[H_i, E_{\vec{\alpha}}] = \alpha_i E_{\vec{\alpha}}, \qquad 1 \le i \le r. \tag{4}$$

From these commutation relations one can give the so called Cartan decomposition of algebra g with respect to the subalgebra h:

$$g \;=\; h \oplus \sum_{\vec{\alpha}\in\Phi} g_{\vec{\alpha}}, \tag{5}$$

where $g_{\vec{\alpha}}$ is one-dimensional vector space, formed by step generator $E_{\vec{\alpha}}$ corresponding to the real r-dimensional vector $\vec{\alpha}$ which is called a root. Φ is a set of all roots.

For each $\vec{\alpha}$ there is one essential step operator $E_{\vec{\alpha}} \in g_{\vec{\alpha}}$ and for $-\vec{\alpha}$ there exist the step operator $E_{-\vec{\alpha}} \in g_{-\vec{\alpha}}$ and

$$E_{-\vec{\alpha}} = E_{\vec{\alpha}}{}^{\star}. \tag{6}$$

It is convenient to form a basis for r-dimensional root space Φ. It is well-known that a basis $\vec{\alpha}_1, \ldots, \vec{\alpha}_r \in \Pi \subset \Phi$ can be chosen in such a way that for any root $\vec{\alpha} \in \Phi$ one can get that

$$\vec{\alpha} = \sum_{i=1}^{i=r} n_i \vec{\alpha}_i, \tag{7}$$

where each $n_i \in Z$ and either $n_i \leq 0$, $1 \leq i \leq r$, or $n_i \geq 0$, $1 \leq i \leq r$. In the former case $\vec{\alpha}$ is said to be positive ($\Phi^+ : \vec{\alpha} \in \Phi^+$) or in the latter case is negative ($\Phi^- : \vec{\alpha} \in \Phi^-$).

Such basis is basis of simple roots.

So if such a basis is constructed one can see that for each $\vec{\alpha} \in \Phi^+ \subset \Phi$, the set of the non-zero roots Φ contains itself $-\vec{\alpha} \in \Phi^- \subset \Phi$, such that

$$\Phi \;=\; \Phi^+ \cup \Phi^-, \qquad \Phi^- \;=\; -\Phi^+. \tag{8}$$

To complete the statement of algebra g we need to consider $[E_{\vec{\alpha}}, E_{\vec{\beta}}]$ for each pair of roots $\vec{\alpha}, \vec{\beta}$. From the Jacobi identity one can get

$$[H_i, [E_{\vec{\alpha}}, E_{\vec{\beta}}]] = (\alpha_i + \beta_i)[E_{\vec{\alpha}}, E_{\vec{\beta}}]. \tag{9}$$

From this one can get

$$\begin{aligned} [E_{\vec{\alpha}}, E_{\vec{\beta}}] &= N_{\vec{\alpha},\vec{\beta}} E_{\vec{\alpha}+\vec{\beta}}, \qquad if \qquad \vec{\alpha}+\vec{\beta} \qquad \in \Phi \\ &= 2\frac{\vec{\alpha}\cdot\vec{H}}{<\vec{\alpha},\vec{\alpha}>}, \qquad if \qquad \vec{\alpha}+\vec{\beta} = 0, \\ &= 0, \qquad\qquad otherwise. \end{aligned} \tag{10}$$

All this choice of generators is called a Cartan-Weyl basis. For each root $\vec{\alpha}$,

$$\{\, E_{\vec{\alpha}}, \qquad 2\,\frac{\vec{\alpha}\cdot\vec{H}}{<\vec{\alpha},\vec{\alpha}>}, \qquad E_{-\vec{\alpha}}\,\} \tag{11}$$

form an $su(2)$ subalgebra, isomorphic to

$$\{I_+, \qquad 2I_3, \qquad I_-\,\}, \tag{12}$$

where

$$[I_+, I_-] = 2I_3, \qquad [I_3, I_\pm] = \pm I_\pm \tag{13}$$

with

$$I_+^* = I_, \qquad\qquad I_3^* = I_3. \tag{14}$$

As consequence one can expect that the eigenvalues of $2\,\frac{\vec{\alpha}\cdot\vec{H}}{<\vec{\alpha},\vec{\alpha}>}$ are integral, i.e.:

$$2\,\frac{<\vec{\alpha},\vec{\beta}>}{<\vec{\alpha},\vec{\alpha}>} \in Z \tag{15}$$

for all roots $\vec{\alpha}$, $\vec{\beta}$.

As the examples one can consider one can consider the root systems for $su(3)$ of rank 2 (see $su(3)$ root system).

Now we introduce the plus-step operators:

$$Q_1 = Q_I^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_2 = Q_{II}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_3 = Q_{III}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{16}$$

and on the minus-step operators:

$$Q_4 = Q_I^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_5 = Q_{II}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad Q_6 = Q_{III}^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{17}$$

We choose the following 3-diagonal operators :

$$H_3 = Q_7 = Q_I^0 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix} \qquad H_8 = Q_8 = Q_{II}^0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_0 = Q_{III}^0 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 \\ 0 & \frac{1}{\sqrt{3}} \\ 0 & 0 \end{pmatrix}$$

For $su(3)$ algebra the positive roots can be chosen as

$$\vec{\alpha}_1 = (1,0), \qquad \vec{\alpha}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2}), \qquad \vec{\alpha}_1 + \vec{\alpha}_2 = \vec{\alpha}_3 = (\frac{1}{2}, \frac{\sqrt{3}}{2}) \tag{19}$$

with

$$< \vec{\alpha}_1, \vec{\alpha}_1 > = < \vec{\alpha}_2, \vec{\alpha}_2 > = 1 \qquad and \qquad < \vec{\alpha}_1, \vec{\alpha}_2 > = -\frac{1}{2}. \tag{20}$$

$$\begin{array}{ll} [\vec{H}, Q_{\pm\vec{\alpha}_1}] = \pm(1,0)\, Q_{\pm\vec{\alpha}_1}; & [Q_{\vec{\alpha}_1}, Q_{-\vec{\alpha}_1}] = 2(1,0) \cdot \vec{H}; \\ [\vec{H}, Q_{\pm\vec{\alpha}_2}] = \pm(-\frac{1}{2}, \frac{\sqrt{3}}{2})\, Q_{\pm\vec{\alpha}_2}, & [Q_{\vec{\alpha}_2}, Q_{-\vec{\alpha}_2}] = 2(-\frac{1}{2}, \frac{\sqrt{3}}{2}) \cdot \vec{H}; \\ [\vec{H}, Q_{\pm\vec{\alpha}_3}] = \pm(\frac{1}{2}, \frac{\sqrt{3}}{2})\, Q_{\pm\vec{\alpha}_3}, & [E_{\vec{\alpha}_3}, E_{-\vec{\alpha}_3}] = 2(\frac{1}{2}, \frac{\sqrt{3}}{2}) \cdot \vec{H}; \end{array} \tag{21}$$

where $\vec{H} = (H_3, H_8)$. The commutation relations of step operators can be also easily written:

$$\begin{array}{ll} [Q_{\vec{\alpha}_1}, Q_{\vec{\alpha}_2}] = Q_{\vec{\alpha}_3}, & [Q_{\vec{\alpha}_1}, Q_{\vec{\alpha}_3}] = [Q_{\vec{\alpha}_2}, Q_{\vec{\alpha}_3}] = 0 \\ [Q_{\vec{\alpha}_1}, Q_{-\vec{\alpha}_3}] = Q_{-\vec{\alpha}_2}, & [Q_{\vec{\alpha}_2}, Q_{-\vec{\alpha}_3}] = Q_{-\vec{\alpha}_1}. \end{array} \tag{22}$$

For A_2 algebra the nonzero roots can be also expressed through the orthonormal basis $\{\vec{e}_i\}, i = 1, 2, 3$, in which all the roots are lying on the plane ortogonal to the vector $\vec{k} = 1 \cdot \vec{e}_1 + 1 \cdot \vec{e}_1 + 1 \cdot \vec{e}_1$, i.e.$\vec{k} \cdot \vec{\alpha} = 0$. Then for this algebra the positive roots are the following:

$$\vec{\alpha}_1 = \vec{e}_1 - \vec{e}_2, \qquad \vec{\alpha}_2 = \vec{e}_2 - \vec{e}_3, \qquad \vec{\alpha}_3 = \vec{e}_1 - \vec{e}_3. \tag{23}$$

This basis can be practically used in general case to give the complete list of simple finite dimensional Lie algebras

$$\begin{array}{ccccc} SU(n) & \pm(e_i - e_j) & 1 \le i \le j \le n & 0 & (n-1) \\ SO(2n) & \pm e_i \pm e_j & 1 \le i \le j \le n & 0 & n \\ SO(2n+1) & \pm e_i \pm e_j & 1 \le i \le j \le n & 0 & n \\ & \pm e_i & 1 \le i \le n & & \\ Sp(n) & \pm e_i \pm e_j & 1 \le i \le j \le n & 0 & n \\ & \pm 2e_i & 1 \le i \le n & & \end{array} \tag{24}$$

Since $su(n)$ is the Lie algebra of traceless $n \times n$ anti-Hermitian matrices there being $(n-1)$ linear independent diagonal matrices. Let $h_{kl} = (e_{kk} - e_{k+1,k+1})$, $k = 1, ..., n-1$ be the choice of the diagonal matrices, and let e_{pq} for $p, q = 1, ..., n$, $p < q$ be the remaining the basis elements:

$$(e_{pq})_{ks} = \delta_{kp}\delta_{sq}. \tag{25}$$

It is easily see that the simple finite-dimensional algebra G can be encoded in the $r \times r$ Cartan matrix

$$A_{ij} = \frac{< \alpha_i, \alpha_j >}{< \alpha_i, \alpha_i >}, \;\; 1 \le i, j \le r, \tag{26}$$

with simple roots, α_i, which generally obeys to the following rules:

$$\begin{array}{rcl} A_{ii} & = & 2 \\ A_{ij} & \le & 0 \\ A_{ij} = 0 & \mapsto & A_{ji} = 0 \\ A_{ij} & \in & Z \qquad = 0, 1, 2, 3 \\ DetA & > & 0. \end{array} \tag{27}$$

The rank of A is equal to r.

$$\begin{aligned} &A_r : Det(A) = (r+1), \\ &D_r : Det(A) = 4, \\ &B_r : Det(A) = 2, \\ &C_r : Det(A) = 2, \end{aligned}$$

$$
\begin{aligned}
&F_4 : Det(A) = 1,\\
&G_2 : Det(A) = 1,\\
&E_r : Det(A) = 9 - r, \qquad r = 6, 7, 8.
\end{aligned}
\tag{28}
$$

Also using theory of the simple roots and Cartan matrices the list of simple killing-Cartan-Lie algebras can be encoded in the Dynkin diagram.

The Dynkin diagram of g is the graph with nodes labeled $1 \ldots, r$ in a bijective correspondence with the set of the simple roots, such that nodes i, j with $i \neq j$ are joined by n_{ij} lines, where $n_{ij} = A_{ij} A_{ji}, i \neq j$.

One can easily get that $A_{ij} A_{ji} = 0, 1, 2, 3.$. Its diagonal elements are equal 2 and its off-diagonal elements are all negative integers or zero. The information in A is codded into Dynkin diagram which is built as follows: it consists of the points for each simple root $\vec{\alpha}_i$ with points $\vec{\alpha}_i$ and $\vec{\alpha}_j$ being joined by $A_{ij} A_{ji}$ lines, with arrow pointing from $\vec{\alpha}_j$ to $\vec{\alpha}_i$ if $< \vec{\alpha}_j, \vec{\alpha}_j >>< \vec{\alpha}_i, \vec{\alpha}_i >$.

Obviously, that $\hat{A}_{ij} = A_{ij}$ for $1 \leq i, j \leq r$, and $\hat{A}_{00} = 2$. For generalized Cartan matrix there are two unique vectors a and $a^\vee$ with positive integer components $(a_0, \ldots, a_r)$ and $(a_0^\vee, \ldots, a_r^\vee)$ with their greatest common divisor equal one, such that

$$
\sum_{i=0}^{r} a_i \hat{A}_{ji} = 0, \qquad \qquad \sum_{i=0}^{r} \hat{A}_{ij} a_j^\vee = 0. \tag{29}
$$

The numbers, a_i and $a_i^\vee$ are called Coxeter and dual Coxeter labels. Sums of the Coxeter and dual Coxeter labels are called by Coxeter h and dual Coxeter numbers $h^\vee$. For symmetric generalized Cartan matrix the both Coxeter labels and numbers coincide. The components a_i, with $i \neq 0$ are just the components of the highest root of Cartan-Lie algebra. The Dynkin diagram for Cartan-Lie algebra can be get from generalized Dynkin diagram of affine algebra by removing one zero node. The generalized Cartan matrices and generalized Dynkin diagrams allow one-to-one to determine affine Kac-Moody algebras .

2 The geometry of ternary generalization of quaternions

Let consider the following construction

$$Q = z_0 + z_1 q_s + z_2 q_s^2, \tag{30}$$

where

$$\begin{aligned} z_0 &= y_0 + q y_1 + q^2 y_2, \\ z_1 &= y_3 + q y_4 + q^2 y_5, \\ z_2 &= y_6 + q x_7 + q^2 y_8 \end{aligned} \tag{31}$$

are the ternary complex numbers and q_s is the new 'imaginary" ternary unit with condition

$$q_s^3 = 1 \tag{32}$$

and

$$q_s q = j q q_s. \tag{33}$$

Then one can see

$$Q = y_0 + q y_1 + q^2 y_2 + y_3 q_s + y_4 q q_s + y_5 q^2 q_s + y_6 q_s^2 + y_7 q q_s^2 + y_8 q^2 q_s^2 \tag{34}$$

We will accept the following notations:

$$\begin{array}{llll} q = q_1, & q_s = q_2, & q^2 q_s^2 = q_3, & (1) \\ q^2 = q_4, & q_s^2 = q_5, & q q_s = q_6, & (2) \\ q q_s^2 = q_7, & q^2 q_s = q_8, & 1 = q_0. & (0) \end{array} \tag{35}$$

Respectively we change the notations of coordinates:

$$\begin{array}{llll} y_1 = x_1, & y_3 = x_2, & y_8 = x_3, & (1) \\ y_2 = x_4, & y_6 = x_5, & y_4 = x_6, & (2) \end{array} \tag{36}$$

In the new notations we have got the following expression:

$$\begin{aligned} Q &= (x_0 + x_7 q_1 q^2 + x_8 q_1^2 q_2) + (x_1 q_1 + x_2 q_2 + x_3 q_1^2 q_2^2) + (x_4 q_1^2 + x_5 q_5^2 + x_6 q_1 q_2) \\ &\equiv z_0(x_0, x_7, x_8) + z_1(x_1, x_2, x_3) + z_2(x_4, x_5, x_6), \end{aligned} \tag{37}$$

where

$$
\begin{aligned}
z_0(a,b,c) &= a + bq_1q_2^2 + cq_1^2q_2 \\
z_1(a,b,c) &= aq_1 + bq_2 + cq_1^2q_2^2 \\
z_2(a,b,c) &= aq_1^2 + bq_2^2 + cq_1q_2
\end{aligned}
\tag{38}
$$

and

$$
\{a,b,c\} = \{x_0,x_7,x_8\},\ \{x_1,x_2,x_3\}, \{x_4,x_5,x_6\}, \tag{39}
$$

with all possible permutations of triples.

It is easily to check:

$-$	$-$	$\{x_0,x_7,x_8\}$	$\{x_1,x_2,x_3\}$	$\{x_4,x_5,x_6\}$
TCl_0	1	$x_0 + x_7q_1q_2^2 + x_8q_1^2q_2$	$x_1q_1 + x_2q_2 + x_3q_1^2q_2^2$	$x_4q_1^2 + x_5q_2^2 + x_6q_1q_2$
	$q_1^2q_2$	$jx_0q_1q_2^2 + j^2x_7q_1^2q_2 + x_8$	$j^2x_1q_1^2q_2^2 + jx_2q_1 + x_3q_2$	$x_4q_2^2 + jx_5q_1q_2 + j^2x_6q_1^2$
	$q_1q_2^2$	$jx_0q_1^2q_2 + x_7 + j^2x_8q_1q_2^2$	$x_1q_2 + jx_2q_1^2q_2^q + j^2x_3q_1$	$j^2x_4q_1q_2 + jx_5q_1^2 + x_6q_2^2$
TCl_1	q_1	$x_0q_1^2 + x_7q_2^2 + x_8q_1q_2$	$x_1 + x_2q_1^2q_2 + x_3q_1q_2^2$	$x_4q_1 + x_5q_1^2q_2^2 + x_6q_2$
	q_2	$x_0q_2^2 + jx_7q_1q_2 + j^2x_8q_1^2$	$jx_1q_1q_2^2 + x_2 + j^2x_3q_1^2q_2$	$j^2x_4q_1^2q_2^2 + x_5q_2 + jx_6q_1$
	$q_1^2q_2^2$	$j^2x_0q_1q_2 + jx_7q_1^2 + x_8q_2^2$	$jx_1q_1^2q_2 + j^2x_2q_1q_2^q + x_3$	$x_4q_2 + j^2x_5q_1 + jx_6q_1^2q_2^2$
TCl_2	q_1^2	$x_0q_1 + x_7q_1^2q_2^2 + x_8q_2$	$x_1q_1^2 + x_2q_1q_2 + x_3q_2^2$	$x_4 + x_5q_1q_2^2 + x_6q_1^2q_2$
	q_2^2	$x_0q_2 + j^2x_7q_1 + jx_8q_1^2q_2^2$	$j^2x_1q_1q_2 + x_2q_2^2 + jx_3q_1^2$	$jx_4q_1^2q_2 + x_5 + j^2x_6q_1q_2^2$
	q_1q_2	$j^2x_0q_1^2q_2^2 + x_7q_2 + jx_8q_1$	$x_1q_2^2 + j^2x_2q_1^2 + jx_3q_1q_2$	$jx_4q_1q_2^2 + j^2x_5q_1^2q_2 + x_6$

(40)

$$
\begin{aligned}
Q &= [z_0(x_0,x_7,x_8) + z_1(x_1,x_2,x_3) + z_2(x_4,x_5,x_6)] \\
&= q_1q_2^2[z_0(x_7,x_8,x_0) + z_1(x_3,x_1,x_2) + z_2(x_5,x_6,x_4)] \\
&= q_1^2q_2[z_0(x_8,x_0,x_7) + z_1(x_2,x_3,x_1) + z_2(x_6,x_4,x_5)]
\end{aligned}
\tag{41}
$$

$$
\begin{aligned}
Q &= q_1[z_0(x_1,x_3,x_2) + z_1(x_4,x_6,x_5) + z_2(x_0,x_7,x_8)] \\
&= q_2[z_0(x_2,x_3,x_1) + z_1(x_5,x_4,x_6) + z_2(x_0,x_8,x_7)] \\
&= q_1^2q_2^2[z_0(x_3,x_2,x_1) + z_1(x_6,x_4,x_5) + z_2(x_0,x_7,x_8)]
\end{aligned}
\tag{42}
$$

$$
\begin{aligned}
Q &= q_1^2[z_0(x_4,x_5,x_6) + z_1(x_0,x_8,x_7) + z_2(x_1,x_3,x_2)] \\
&= q_2^2[z_0(x_5,x_6,x_4) + z_1(x_7,x_0,x_8) + z_2(x_3,x_2,x_1)] \\
&= q_1q_2[z_0(x_6,x_4,x_5) + z_1(x_8,x_7,x_0) + z_2(x_2,x_1,x_3)]
\end{aligned}
\tag{43}
$$

Now we can rewrite the expression for Q, $\tilde{Q}$, and $\tilde{\tilde{Q}}$ in the following way:

$$\begin{array}{rcl}
Q & = & z_2(x_4,x_5,x_6)+q_1 z_2(x_0,x_7,x_8)+q_1^2 z_2(x_1,x_3,x_2) \\
\tilde{Q} & = & \tilde{z}_2(x_4,x_5,x_6)+jq_1\tilde{z}_2(x_0,x_7,x_8)+j^2q_1^2\tilde{z}_2(x_1,x_3,x_2) \\
\tilde{\tilde{Q}} & = & \tilde{\tilde{z}}_2(x_4,x_5,x_6)+j^2q_1\tilde{\tilde{z}}_2(x_0,x_7,x_8)+jq_1^2\tilde{\tilde{z}}_2(x_1,x_3,x_2)
\end{array} \tag{44}$$

where we accept that

$$\begin{array}{ll}
\tilde{q}_1 = jq_1, & \tilde{q}_2 = jq_2, \\
\tilde{\tilde{q}}_1 = j^2q_1, & \tilde{\tilde{q}}_2.
\end{array} \tag{45}$$

We would like to calculate the product $Q\tilde{Q}\tilde{\tilde{Q}}$ what in general contains itself $9\times 9\times 9 = 729$ terms, *i.e.*

$$Q\times\tilde{Q}\times\tilde{\tilde{Q}} = A_0(x_0,...,x_8)q_0 + A_1(x_0,...,x_8)q_1 + \ldots + A_8(x_0,...,x_8)q_8 \tag{46}$$

In general in this product one can meet inside A_p ($p = 0,1,...,8$) the following term structures:

$$\begin{array}{ll}
x_p^3, & p = 0,1,...,8 \\
x_p^2 x_k, & p \neq r \\
x_p x_k x_l, & p \neq k \neq l.
\end{array} \tag{47}$$

For this expansion one can easily see that

$$729 = 9\,(x_p^3 - terms)\; +\; 72\times 3\,(x_p^2x_k - terms) + 84\times 6\,(x_px_kx_l - terms). \tag{48}$$

Note that we would like to save just terms proportional to $q_0 = 1$ - terms, *i.e.* to find the A_0 magnitude. All others must be equal to zero. Since $q_p^3 = 1$ for $p = 0,1,...,8$ A_0 contains the first nine pure cubic terms.

We can see how in the product $Q\tilde{Q}\tilde{\tilde{Q}}$ vanish the terms $72\times 3\,(x_p^2x_k - terms$. From the expression

respectively. Now one can see that all terms disappear. In this product we can find the nonvanishing terms proportional $q_0 = 1$:

$$\begin{array}{l}
x_0^3 + x_7^3 + x_8^3 - 3x_0x_7x_8, \\
x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3, \\
x_0^4 + x_5^3 + x_6^3 - 3x_4x_5x_6,
\end{array} \tag{49}$$

where we took into account that

$$\begin{aligned}
q_0 q_7 q_8 &\sim 1 \\
q_1 q_2 q_3 &\sim 1 \\
q_4 q_5 q_6 &\sim 1.
\end{aligned} \tag{50}$$

Also, one can also find the other combination proportional to 1:

$$\begin{aligned}
q_0(q_1 q_4 + q_2 q_5 + q_3 q_6) &\sim 1 \\
q_7(q_1 q_5 + q_2 q_6 + q_3 q_1) &\sim 1 \\
q_8(q_1 q_6 + q_2 q_4 + q_3 q_5) &\sim 1.
\end{aligned} \tag{51}$$

So, we have got in the triple product the

$$9\,(\,x_p^3\,)\;+\;12\times 6\,(\,x_p x_k x_l\,)\;=\;81\,(\,terms\,). \tag{52}$$

The $729 - 81 = 72 \times 3 + 72 \times 6 = 648$ terms are vanished.

Thus, we expect to get the equation for the unit ternary "quaternion" surface in the following form:

$$\begin{aligned}
& x_0^3 + x_7^3 + x_8^3 - 3x_0 x_7 x_8 \\
& x_1^3 + x_2^3 + x_3^3 - 3x_1 x_2 x_3 \\
& x_0^4 + x_5^3 + x_6^3 - 3x_4 x_5 x_6 \\
& -3x_0(x_1 x_4 + x_2 x_5 + x_3 x_6) \\
& -3x_7(x_1 x_5 + x_2 x_6 + x_3 x_4) \\
& -3x_8(x_1 x_6 + x_2 x_4 + x_3 x_5) = 1
\end{aligned} \tag{53}$$

In this product we can find the nonvanishing terms proportional $q_0 = 1$:

$$\begin{aligned}
& x_0^3 + x_7^3 + x_8^3 - 3x_0 x_7 x_8, \\
& x_1^3 + x_2^3 + x_3^3 - 3x_1 x_2 x_3, \\
& x_0^4 + x_5^3 + x_6^3 - 3x_4 x_5 x_6,
\end{aligned} \tag{54}$$

where we took into account that

$$\begin{aligned}
q_0 q_7 q_8 &\sim 1 \\
q_1 q_2 q_3 &\sim 1 \\
q_4 q_5 q_6 &\sim 1.
\end{aligned} \tag{55}$$

Also, one can also find the other combination proportional to 1:

$$\begin{aligned}
&q_0(q_1q_4 + q_2q_5 + q_3q_6) \sim 1 \\
&q_7(q_1q_5 + q_2q_6 + q_3q_1) \sim 1 \\
&q_8(q_1q_6 + q_2q_4 + q_3q_5) \sim 1.
\end{aligned} \tag{56}$$

$$\begin{aligned}
Q &= (x_0 + x_7q_7 + x_8q_8) + (x_1q_1 + x_2q_2 + x_3q_3) + (x_4q_4 + x_5q_5 + x_6q_6) \\
\tilde{Q} &= (x_0 + jx_7q_7 + j^2x_8q_8) + j(x_1q_1 + x_2q_2 + x_3q_3) + j^2(x_4q_4 + x_5q_5 + x_6q_6) \\
\tilde{\tilde{Q}} &= (x_0 + j^2x_7q_7 + jx_8q_8) + j^2(x_1q_1 + x_2q_2 + x_3q_3) + j(x_4q_4 + x_5q_5 + x_6q_6)
\end{aligned} \tag{57}$$

$$Q_1 = q\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \tag{58}$$

$$Q_2 = q^2\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \tag{59}$$

$$Q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \tag{60}$$

$$Q_4 = Q_1^2 = q^2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{61}$$

$$Q_5 = Q_2^2 = q\begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \tag{62}$$

$$Q_6 = Q_3^2 = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}. \tag{63}$$

$$Q_7 = \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{64}$$

$$Q_8 = \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{65}$$

$$Q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{66}$$

$$\begin{array}{l} Q_1Q_2 = j^2Q_6,\ Q_2Q_3 = j^2q^2Q_4,\ Q_3Q_1 = j^2qQ_5 \\ Q_2Q_1 = jQ_6,\ Q_3Q_2 = jq^2Q_4,\ Q_1Q_3 = jqQ_5 \end{array} \tag{67}$$

$$\begin{array}{l} Q_4Q_5 = j^2Q_3,\ Q_5Q_6 = j^2qQ_1,\ Q_6Q_4 = j^2q^2Q_2 \\ Q_5Q_4 = jQ_3,\ Q_6Q_5 = jqQ_1,\ Q_4Q_6 = jq^2Q_2 \end{array} \tag{68}$$

$$Q_1Q_5 = q^2j \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_5Q_1 = q^2j^2 \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{69}$$

$$Q_1Q_6 = qj^2 \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_6Q_1 = qj \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{70}$$

$$Q_2Q_4 = qj^2 \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_4Q_2 = qj \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{71}$$

$$Q_2Q_6 = q^2j \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_6Q_2 = q^2j^2 \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{72}$$

$$Q_3Q_4 = q^2 j \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_4Q_3 = q^2 j^2 \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{73}$$

$$Q_3Q_5 = qj^2 \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad Q_5Q_3 = qj \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{74}$$

$$\begin{array}{l} Q_1Q_4 = Q_0,\ Q_1Q_5 = q^2jQ_8,\ Q_1Q_6 = qj^2Q_7 \\ Q_4Q_1 = Q_0,\ Q_5Q_1 = q^2j^2Q_8,\ Q_6Q_1 = jqQ_7 \end{array} \tag{75}$$

$$\begin{array}{l} Q_2Q_4 = qj^2Q_7,\ Q_2Q_5 = Q_0,\ Q_2Q_6 = j^2q^2Q_7 \\ Q_4Q_2 = qjQ_7,\ Q_5Q_2 = Q_0,\ Q_6Q_2 = jq^2Q_7 \end{array} \tag{76}$$

$$\begin{array}{l} Q_3Q_4 = q^2jQ_8,\ Q_3Q_5 = qj^2Q_7,\ Q_3Q_6 = Q_0 \\ Q_4Q_3 = q^2j^2jQ_8,\ Q_5Q_3 = qjQ_7,\ Q_6Q_3 = Q_0 \end{array} \tag{77}$$

The ternary conjugation include two operations:

- 1. $\tilde{q} = jq$;

- 2. $\{1 \to 2, 2 \to 3, 3 \to 1\}$.

Let us check the second operation. For this consider two 3×3 matrices:

$$A = \begin{pmatrix} a_1 & b_1 & c_1 \\ c_2 & a_2 & b_2 \\ b_3 & c_3 & a_3 \end{pmatrix}, \qquad and \qquad B = \begin{pmatrix} u_1 & v_1 & w_1 \\ w_2 & u_2 & v_2 \\ v_3 & w_3 & u_3 \end{pmatrix} \tag{78}$$

Then

$$\tilde{A} = \begin{pmatrix} a_3 & b_3 & c_3 \\ c_1 & a_1 & b_1 \\ b_2 & c_2 & a_2 \end{pmatrix}, \qquad and \qquad \tilde{B} = \begin{pmatrix} u_3 & v_3 & w_3 \\ w_1 & u_1 & v_1 \\ v_2 & w_2 & u_2 \end{pmatrix} \tag{79}$$

respectively.

Take the product of these two matrices in both cases:

$$C = A\cdot B = \begin{pmatrix} a_1u_1+b_1w_2+c_1v_3 & a_1v_1+b_1u_2+c_1w_3 & a_1w_1+b_1v_2+c_1u_3 \\ c_2u_1+a_2w_2+b_2v_3 & c_2v_1+a_2u_2+b_2w_3 & c_2w_1+a_2v_2+b_2u_3 \\ b_3u_1+c_3w_2+a_3v_3 & b_3v_1+c_3u_2+a_3w_3 & b_3w_1+c_3v_2+a_3u_3 \end{pmatrix}, \tag{80}$$

$$\tilde{A}\cdot\tilde{B} = \begin{pmatrix} b_3w_1+c_3v_2+a_3u_3 & b_3u_1+c_3w_2+a_3v_3 & b_3v_1+c_3u_2+a_3w_3 \\ a_1w_1+b_1v_2+c_1u_3 & a_1u_1+b_1w_2+c_1v_3 & a_1v_1+b_1u_2+c_1w_3 \\ c_2w_1+a_2v_2+b_2u_3 & c_2u_1+a_2w_2+b_2v_3 & c_2v_1+a_2u_2+b_2w_3 \end{pmatrix}, \tag{81}$$

Compare the last expression with the expressio0n of $\tilde{C}$ one can see that:

$$(A\tilde{\cdot}B) = \tilde{A}\cdot\tilde{B}. \tag{82}$$

$$\tilde{Q}_1 = jq\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} = jQ_1, \tag{83}$$

$$\tilde{Q}_2 = j^2q^2\begin{pmatrix} 0 & j^2 & 0 \\ 0 & 0 & 1 \\ j & 0 & 0 \end{pmatrix} = jQ_2, \tag{84}$$

$$\tilde{Q}_3 = \begin{pmatrix} 0 & j & 0 \\ 0 & 0 & 1 \\ j^2 & 0 & 0 \end{pmatrix} = jQ_3, \tag{85}$$

$$\tilde{Q}_4 = j^2q^2\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = j^2Q_4, \tag{86}$$

$$\tilde{Q}_5 = jq\begin{pmatrix} 0 & 0 & j^2 \\ j & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = j^2Q_5, \tag{87}$$

$$\tilde{Q}_6 = \begin{pmatrix} 0 & 0 & j \\ j^2 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = j^2Q_6 \tag{88}$$

$$Q_7 = j \begin{pmatrix} j & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{89}$$

$$Q_8 = j^2 \begin{pmatrix} j^2 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{90}$$

$$\begin{array}{lll} Q_7Q_1 = qQ_2, & Q_7Q_2 = qQ_3, & Q_7Q_3 = qQ_1 \\ Q_8Q_2 = q^2Q_1, & Q_8Q_3 = q^2Q_2, & Q_8Q_1 = q^2Q_3. \end{array} \tag{91}$$

$$\begin{array}{lll} Q_7Q_4 = qQ_6, & Q_7Q_6 = qQ_5, & Q_7Q_5 = qQ_4 \\ Q_8Q_4 = q^2Q_5, & Q_8Q_5 = q^2Q_6, & Q_8Q_6 = q^2Q_4. \end{array} \tag{92}$$

$$q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{93}$$

3 Ternary TU(3)-algebra

We can consider the 3×3 matrix realization of $q-$ algebra:

$$\begin{array}{c} q_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \, q_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \, q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix}, \\ q_4 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \, q_5 = \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \, q_6 = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix}, \\ q_7 = j \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \, q_8 = j^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, \, q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array} \tag{94}$$

which satisfy to the ternary algebra:

$$\{A, B, C\}_{S_3} = ABC + BCA + CAB - BAC - ACB - CBA. \tag{95}$$

Here $j = \exp(2\mathbf{i}\pi/3)$ and S_3 is the permutation group of three elements.

On the next table we can give the ternary commutation relations for the matrices q_k:

$$\{q_k, q_l, q_m\}_{S_3} = f^n_{klm} q_n. \tag{96}$$

One can check that each triple commutator $\{q_k, q_l, q_m\}$, defined by triple numbers, $\{klm\}$ with $k, l, m = 0, 1, 2, ..., 8$, gives just one matrix q_n with the corresponding coefficient f^n_{klm} giving in the table:

The q_k elements satisfy to the ternary algebra:

$$\{A, B, C\}_{S_3} = ABC + BCA + CAB - BAC - ACB - CBA. \tag{97}$$

Here $j = \exp(2\mathbf{i}\pi/3)$ and S_3 is the permutation group of three elements.

On the next table we can give the ternary commutation relations for the matrices q_k:

$$\{q_k, q_l, q_m\}_{S_3} = f^n_{klm} q_n. \tag{98}$$

One can check that each triple commutator $\{q_k, q_l, q_m\}$, defined by triple numbers, $\{klm\}$ with $k, l, m = 0, 1, 2, ..., 8$, gives just one matrix q_n with the corresponding coefficient f^n_{klm} giving in the table:

One can find $C^2_9 = 84$ ternary commutation relations. But there one can see that there are also $C^2_8 = 28$commutation relations which correspond to the $su(3)$ algebra! Therefore, it is naturally to represent the q-numbers as ternary generalization of quaternions. If one can take from S_3 commutation relations $C = q_0$ the commutation relations naturally are going to S_2 Lie commutation relations:

$$\{q_a, q_b, q_0\}_{S_3} = q_a q_b q_0 + q_b q_0 q_a + q_0 q_a q_b - q_b q_a q_0 - q_a q_0 q_b - q_0 q_b q_a = q_a q_b - q_b q_a, \tag{99}$$

where $a \neq b \neq 0$. On the table such 28- cases one can see $\{kl0\}$.

We can consider the 3×3 matrix realization of $q-$ algebra:

$$q_1 = q \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad q_2 = q^2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j \\ j^2 & 0 & 0 \end{pmatrix}, \quad q_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & j^2 \\ j & 0 & 0 \end{pmatrix},$$
$$q_4 = q^2 \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad q_5 = q \begin{pmatrix} 0 & 0 & j \\ 1 & 0 & 0 \\ 0 & j^2 & 0 \end{pmatrix}, \quad q_6 = \begin{pmatrix} 0 & 0 & j^2 \\ 1 & 0 & 0 \\ 0 & j & 0 \end{pmatrix},$$
$$q_7 = q^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & j^2 \end{pmatrix}, \quad q_8 = q \begin{pmatrix} 1 & 0 & 0 \\ 0 & j^2 & 0 \\ 0 & 0 & j \end{pmatrix}, \quad q_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{100}$$

Table 1: *The ternary commutation relations*

N	$\{klm\} \to \{n\}$	f^n_{klm}	N	$\{klm\} \to \{n\}$	f^n_{klm}	N	$\{klm\} \to \{n\}$	f^n_{klm}
1	$\{123\} \to \{0\}$	$3(j^2-j)$	2	$\{124\} \to \{2\}$	$j(1-j)$	3	$\{125\} \to \{1\}$	$2(j^2-j)$
4	$\{126\} \to \{3\}$	$j(1-j)$	5	$\{127\} \to \{5\}$	$2(1-j)$	6	$\{128\} \to \{4\}$	$2(j^2-1)$
7	$\{120\} \to \{6\}$	(j^2-j)	8	$\{134\} \to \{3\}$	(j^2-j)	9	$\{135\} \to \{2\}$	$2(j-j^2)$
10	$\{136\} \to \{1\}$	(j^2-j)	11	$\{137\} \to \{4\}$	$2(j-1)$	12	$\{138\} \to \{6\}$	$2(1-j^2)$
13	$\{130\} \to \{5\}$	$(j-j^2)$	14	$\{145\} \to \{5\}$	$(j-j^2)$	15	$\{146\} \to \{6\}$	(j^2-j)
16	$\{147\} \to \{7\}$	(j^2-j)	17	$\{148\} \to \{8\}$	$(j-j^2)$	18	$\{140\} \to \tilde{O}$	0
19	$\{156\} \to \{4\}$	$2j(j-1)$	20	$\{157\} \to \{0\}$	$3(1-j)$	21	$\{158\} \to \{7\}$	$2(1-j)$
22	$\{150\} \to \{8\}$	$(1-j)$	23	$\{167\} \to \{8\}$	$2(1-j^2)$	24	$\{168\} \to \{0\}$	$3(1-j^2)$
25	$\{160\} \to \{7\}$	$(1-j^2)$	26	$\{178\} \to \{1\}$	$(j-j^2)$	27	$\{170\} \to \{2\}$	$(j-1)$
28	$\{180\} \to \{3\}$	(j^2-1)	29	$\{234\} \to \{1\}$	$2(j^2-j)$	30	$\{235\} \to \{3\}$	$(j-j^2)$
31	$\{236\} \to \{2\}$	$(j-j^2)$	32	$\{237\} \to \{6\}$	$2(1-j)$	33	$\{238\} \to \{5\}$	$2(j^2-1)$
34	$\{230\} \to \{4\}$	(j^2-j)	35	$\{245\} \to \{4\}$	$(j-j^2)$	36	$\{246\} \to \{5\}$	$2(j-j^2)$
37	$\{247\} \to \{8\}$	$2(1-j^2)$	38	$\{248\} \to \{0\}$	$3(1-j^2)$	39	$\{240\} \to \{7\}$	$(1-j^2)$
40	$\{256\} \to \{6\}$	$(j-j^2)$	41	$\{257\} \to \{7\}$	(j^2-j)	42	$\{258\} \to \{8\}$	$(j-j^2)$
43	$\{250\} \to \tilde{O}$	0	44	$\{267\} \to \{0\}$	$3(1-j)$	45	$\{268\} \to \{7\}$	$2(1-j)$
46	$\{260\} \to \{8\}$	$(1-j)$	47	$\{278\} \to \{2\}$	$(j-j^2)$	48	$\{270\} \to \{3\}$	$(j-1)$
49	$\{280\} \to \{1\}$	(j^2-1)	50	$\{345\} \to \{6\}$	$2(j^2-j)$	51	$\{346\} \to \{4\}$	(j^2-j)
52	$\{347\} \to \{0\}$	$3(1-j)$	53	$\{348\} \to \{7\}$	$2(1-j)$	54	$\{340\} \to \{8\}$	$(1-j)$
55	$\{356\} \to \{5\}$	$j-j^2$	56	$\{357\} \to \{8\}$	$2(1-j^2)$	57	$\{358\} \to \{0\}$	$3(1-j^2)$
58	$\{350\} \to \{7\}$	$(1-j^2)$	59	$\{367\} \to \{7\}$	(j^2-j)	60	$\{368\} \to \{8\}$	$(j-j^2)$
61	$\{360\} \to \tilde{O}$	0	62	$\{378\} \to \{3\}$	$(j-j^2)$	63	$\{370\} \to \{1\}$	$(j-1)$
64	$\{380\} \to \{2\}$	(j^2-1)	65	$\{456\} \to \{0\}$	$3(j^2-j)$	66	$\{457\} \to \{1\}$	$2(1-j)$
67	$\{458\} \to \{2\}$	$2(j^2-1)$	68	$\{450\} \to \{3\}$	(j^2-j)	69	$\{467\} \to \{1\}$	$2(j-1)$
70	$\{468\} \to \{1\}$	$2(1-j^2)$	71	$\{460\} \to \{2\}$	$(j-j^2)$	72	$\{478\} \to \{4\}$	(j^2-j)
73	$\{470\} \to \{6\}$	$(1-j)$	74	$\{480\} \to \{5\}$	$(1-j^2)$	75	$\{567\} \to \{2\}$	$2(1-j)$
76	$\{568\} \to \{3\}$	$2(j^2-j)$	77	$\{560\} \to \{1\}$	(j^2-j)	78	$\{578\} \to \{5\}$	(j^2-j)
79	$\{570\} \to \{4\}$	$(1-j)$	80	$\{580\} \to \{6\}$	$(1-j^2)$	81	$\{678\} \to \{6\}$	(j^2-j)
82	$\{670\} \to \{5\}$	$(1-j)$	83	$\{680\} \to \{4\}$	$(1-j^2)$	84	$\{780\} \to \tilde{O}$	0

which satisfy to the ternary algebra:

$$\{A,B,C\}_{S_3} = ABC + BCA + CAB - BAC - ACB - CBA. \tag{101}$$

Here $j = \exp(2\mathbf{i}\pi/3)$ and S_3 is the permutation group of three elements.

On the next table we can give the ternary commutation relations for the matrices q_k:

$$\{q_k, q_l, q_m\}_{S_3} = f^n_{klm} q_n. \tag{102}$$

One can check that each triple commutator $\{q_k, q_l, q_m\}$, defined by triple numbers, $\{klm\}$ with $k, l, m = 0, 1, 2, ..., 8$, gives just one matrix q_n with the corresponding coefficient f^n_{klm} giving in the table:

4 The geometrical representations of ternary "quaternions"

Let us define the following product:

$$\begin{aligned} \hat{Q} = \sum_{a=0}^{a=8} \{x_a q_a\} &= \begin{pmatrix} x_0 + jx_7 + j^2 x_8 & x_1 + x_2 + x_3 & x_4 + jx_5 + j^2 x_6 \\ x_4 + x_5 + x_6 & x_0 + j^2 x_7 + jx_8 & x_1 + jx_2 + j^2 x_3 \\ x_1 + j^2 x_2 + jx_3 & x_4 + j^2 x_5 + jx_6 & x_0 + x_7 + x_8 \end{pmatrix} \\ &= \begin{pmatrix} \tilde{z}_0 & z_1 & \tilde{z}_2 \\ z_2 & \tilde{\tilde{z}}_0 & \tilde{z}_1 \\ \tilde{\tilde{z}}_1 & \tilde{\tilde{z}}_2 & z_0 \end{pmatrix}. \end{aligned} \tag{103}$$

$$\begin{aligned} & (z_0 z_1 z_2 + \tilde{z}_0 \tilde{z}_1 \tilde{z}_2 + \tilde{\tilde{z}}_0 \tilde{\tilde{z}}_1 \tilde{\tilde{z}}_2) \\ = & \ [(x_0 + x_7 + x_8)(x_1 + x_2 + x_3)(x_4 + x_5 + x_6)] \\ + & \ [(x_0 + jx_7 + j^2 x_8)(x_1 + jx_2 + j^2 x_3)(x_4 + j^2 x_5 + jx_6)] \\ + & \ [(x_0 + j^2 x_7 + jx_8)(x_1 + j^2 x_2 + jx_3)(x_4 + jx_5 + j^2 x_6)] \\ = & \ [(x_0 + x_7 + x_8) \\ & \cdot(x_1x_4 + x_1x_5 + x_1x_6 + x_2x_4 + x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_6] \\ + & \ [(x_0 + jx_7 + j^2 x_8) \\ & \cdot(x_1x_4 + j^2 x_1x_5 + jx_1x_6 + jx_2x_4 + x_2x_5 + j^2 x_2x_6 + j^2 x_3x_4 + jx_3x_5 + x_3x_6] \\ + & \ [(x_0 + j^2 x_7 + jx_8) \\ & \cdot(x_1x_4 + jx_1x_5 + j^2 x_1x_6 + j^2 x_2x_4 + x_2x_5 + jx_2x_6 + jx_3x_4 + j^2 x_3x_5 + x_3x_6] \\ = & \ \{3x_0[x_1x_4 + x_2x_5 + x_3x_6]\} \\ + & \ \{3x_7[x_1x_5 + x_2x_6 + x_3x_4]\} \\ + & \ \{3x_8[x_1x_6 + x_2x_4 + x_3x_5]\} \end{aligned}$$

Then we can define the norm of the ternary quaternion through the determinant

$$\begin{aligned}
Det\hat{Q} &= [(x_0 + jx_7 + j^2x_8)(x_0 + j^2x_7 + jx_8)(x_0 + x_7 + x_8)] \\
&+ [(x_1 + x_2 + x_3)(x_1 + jx_2 + j^2x_3)(x_1 + j^2x_2 + jx_3)] \\
&+ [(x_4 + x_5 + x_6)(x_4 + jx_5 + j^2x_6)(x_4 + j^2x_5 + jx_6)] \\
&- \{(x_0 + j^2x_7 + jx_8)(x_1 + j^2x_2 + jx_3)(x_4 + jx_5 + j^2x_6) \\
&- (x_0 + jx_7 + j^2x_8)(x_1 + jx_2 + j^2x_3)(x_4 + j^2x_5 + jx_6) \\
&- (x_0 + x_7 + x_8)(x_1 + x_2 + x_3)(x_4 + x_5 + x_6)\} \\
&= |z_0|^3 + |z_1|^3 + |z_2|^3 - (z_0z_1z_2 + \tilde{z}_0\tilde{z}_1\tilde{z}_2 + \tilde{\tilde{z}}_0\tilde{\tilde{z}}_1\tilde{\tilde{z}}_2)
\end{aligned} \tag{104}$$

$$\begin{array}{lll}
z_0 = x_0 + x_7q + x_8q^2 & \tilde{z}_0 = x_0 + jx_7q + j^2x_8q^2 & \tilde{\tilde{z}}_0 = x_0 + j^2x_7q + jx_8q^2 \\
z_1 = x_1 + x_2q + x_3q^2 & \tilde{z}_1 = x_1 + jx_2q + j^2x_3q^2 & \tilde{\tilde{z}}_1 = x_1 + j^2x_2q + jx_3q^2 \\
z_2 = x_4 + x_5q + x_6q^2 & \tilde{z}_2 = x_4 + jx_5q + j^2x_6q^2 & \tilde{\tilde{z}}_2 = x_4 + j^2x_5q + jx_6q^2
\end{array} \tag{105}$$

or

$$\begin{aligned}
Det\hat{Q} &= [x_0^3 + x_7^3 + x_8^3 - 3x_0x_7x_8] \\
&+ [x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3] \\
&+ [(x_4^3 + x_5^3 + x_6^3 - 3x_4x_5x_6)] \\
&- \{(x_0 + j^2x_7 + jx_8) \\
&\cdot [x_1x_4 + jx_1x_5 + j^2x_1x_6 + j^2x_2x_4 + x_2x_5 + jx_2x_6 + jx_3x_4 + j^2x_3x_5 + x_3x_6]\} \\
&- \{(x_0 + x_7 + x_8) \\
&\cdot [x_1x_4 + j^2x_1x_5 + jx_1x_6 + jx_2x_4 + x_2x_5 + j^2x_2x_6 + j^2x_3x_4 + jx_3x_5 + x_3x_6]\} \\
&- \{(x_0 + jx_7 + j^2x_8) \\
&\cdot [x_1x_4 + x_1x_5 + x_1x_6 + x_2x_4 + x_2x_5 + x_2x_6 + x_3x_4 + x_3x_5 + x_3x_6]\}
\end{aligned} \tag{106}$$

$$\begin{array}{lll}
z_0 = x_0 + x_7q + x_8q^2 & \tilde{z}_0 = x_0 + jx_7q + j^2x_8q^2 & \tilde{\tilde{z}}_0 = x_0 + j^2x_7q + jx_8q^2 \\
z_1 = x_1 + x_2q + x_3q^2 & \tilde{z}_1 = x_1 + jx_2q + j^2x_3q^2 & \tilde{\tilde{z}}_1 = x_1 + j^2x_2q + jx_3q^2 \\
z_2 = x_4 + x_5q + x_6q^2 & \tilde{z}_2 = x_4 + jx_5q + j^2x_6q^2 & \tilde{\tilde{z}}_2 = x_4 + j^2x_5q + jx_6q^2
\end{array} \tag{107}$$

or(1)

$$\begin{aligned}
Det\hat{Q} &= [x_0^3 + x_7^3 + x_8^3 - 3x_0x_7x_8] \\
&+ [x_1^3 + x_2^3 + x_3^3 - 3x_1x_2x_3] \\
&+ [(x_4^3 + x_5^3 + x_6^3 - 3x_4x_5x_6)] \\
&- \{3x_0[x_1x_4 + x_2x_5 + x_3x_6]\} \\
&- \{3x_7[x_1x_5 + x_2x_6 + x_3x_4]\} \\
&- \{3x_8[x_1x_6 + x_2x_4 + x_3x_5]\}
\end{aligned} \tag{108}$$

or (2)

$$\begin{aligned} Det\hat{Q} &= [x_0^3 + x_1^3 + x_4^3 - 3x_0x_1x_4] \\ &+ [x_7^3 + x_2^3 + x_6^3 - 3x_7x_2x_6] \\ &+ [(x_8^3 + x_5^3 + x_3^3 - 3x_8x_5x_3)] \\ &- \{3x_0[x_7x_8 + x_2x_5 + x_3x_6]\} \\ &- \{3x_1[x_2x_3 + x_5x_7 + x_6x_8]\} \\ &- \{3x_4[x_5x_6 + x_3x_7 + x_2x_8]\} \end{aligned} \tag{109}$$

$$\begin{array}{lll} z_0 = x_0 + x_1q + x_4q^2 & \tilde{z}_0 = x_0 + jx_1q + j^2x_4^2 & \tilde{\tilde{z}}_0 = x_0 + j^2x_1q + jx_4q^2 \\ z_1 = x_7 + x_2q + x_6q^2 & \tilde{z}_1 = x_7 + jx_2q + j^2x_6q^2 & \tilde{\tilde{z}}_1 = x_7 + j^2x_2q + jx_6q^2 \\ z_2 = x_8 + x_5q + x_3q^2 & \tilde{z}_2 = x_8 + jx_5q + j^2x_3q^2 & \tilde{\tilde{z}}_2 = x_8 + j^2x_5q + jx_3q^2 \end{array} \tag{110}$$

or (3)

$$\begin{aligned} Det\hat{Q} &= [x_0^3 + x_2^3 + x_5^3 - 3x_0x_2x_5] \\ &+ [x_7^3 + x_3^3 + x_4^3 - 3x_7x_3x_4] \\ &+ [(x_8^3 + x_1^3 + x_6^3 - 3x_8x_1x_6)] \\ &- \{3x_0[x_7x_8 + x_1x_4 + x_3x_6]\} \\ &- \{3x_2[x_1x_3 + x_6x_7 + x_4x_8]\} \\ &- \{3x_5[x_3x_4 + x_1x_7 + x_3x_8]\} \end{aligned} \tag{111}$$

$$\begin{array}{lll} z_0 = x_0 + x_2q + x_5q^2 & \tilde{z}_0 = x_0 + jx_2q + j^2x_5q^2 & \tilde{\tilde{z}}_0 = x_0 + j^2x_2q + jx_5q^2 \\ z_1 = x_7 + x_3q + x_4q^2 & \tilde{z}_1 = x_7 + jx_3q + j^2x_4q^2 & \tilde{\tilde{z}}_1 = x_7 + j^2x_3q + jx_4q^2 \\ z_2 = x_8 + x_1q + x_6q^2 & \tilde{z}_2 = x_8 + jx_1q + j^2x_6q^2 & \tilde{\tilde{z}}_2 = x_8 + j^2x_1q + jx_6q^2 \end{array} \tag{112}$$

or (4)

$$\begin{aligned} Det\hat{Q} &= [x_0^3 + x_3^3 + x_6^3 - 3x_0x_3x_6] \\ &+ [x_7^3 + x_1^3 + x_5^3 - 3x_1x_5x_7] \\ &+ [(x_8^3 + x_2^3 + x_4^3 - 3x_2x_4x_8)] \\ &- \{3x_0[x_7x_8 + x_1x_4 + x_2x_5]\} \\ &- \{3x_3[x_1x_2 + x_4x_7 + x_5x_8]\} \\ &- \{3x_6[x_4x_5 + x_2x_7 + x_1x_8]\} \end{aligned} \tag{113}$$

$$\begin{array}{lll} z_0 = x_0 + x_3 q + x_6 q^2 & \tilde{z}_0 = x_0 + j x_3 q + j^2 x_6 q^2 & \tilde{\tilde{z}}_0 = x_0 + j^2 x_3 q + j x_6 q^2 \\ z_1 = x_7 + x_1 q + x_5 q^2 & \tilde{z}_1 = x_7 + j x_1 q + j^2 x_5 q^2 & \tilde{\tilde{z}}_1 = x_7 + j^2 x_1 q + j x_5 q^2 \\ z_2 = x_8 + x_2 q + x_4 q^2 & \tilde{z}_2 = x_8 + j x_2 q + j^2 x_4 q^2 & \tilde{\tilde{z}}_2 = x_8 + j^2 x_2 q + j x_4 q^2 \end{array} \tag{114}$$

$$\begin{array}{ccc} 0 & 8 & 7 \\ 1 & 2 & 3 \\ 4 & 6 & 5 \end{array} \qquad \begin{array}{ccc} 0 & 8 & 7 \\ 2 & 3 & 1 \\ 5 & 4 & 6 \end{array} \qquad \begin{array}{ccc} 0 & 8 & 7 \\ 3 & 1 & 2 \\ 6 & 5 & 4 \end{array} \tag{115}$$

$$\begin{array}{ccc} 0 & 1 & 4 \\ 2 & 7 & 6 \\ 5 & 3 & 8 \end{array} \qquad \begin{array}{ccc} 0 & 1 & 4 \\ 6 & 2 & 7 \\ 3 & 8 & 5 \end{array} \qquad \begin{array}{ccc} 0 & 1 & 4 \\ 7 & 6 & 2 \\ 8 & 5 & 3 \end{array} \tag{116}$$

$$\begin{array}{ccc} 0 & 2 & 5 \\ 3 & 7 & 4 \\ 6 & 1 & 8 \end{array} \qquad \begin{array}{ccc} 0 & 2 & 5 \\ 4 & 3 & 7 \\ 1 & 8 & 6 \end{array} \qquad \begin{array}{ccc} 0 & 2 & 5 \\ 7 & 4 & 3 \\ 8 & 6 & 1 \end{array} \tag{117}$$

$$\begin{array}{ccc} 0 & 3 & 6 \\ 1 & 7 & 5 \\ 4 & 2 & 8 \end{array} \qquad \begin{array}{ccc} 0 & 3 & 6 \\ 5 & 1 & 7 \\ 2 & 8 & 4 \end{array} \qquad \begin{array}{ccc} 0 & 3 & 6 \\ 7 & 5 & 1 \\ 8 & 4 & 2 \end{array} \tag{118}$$

One can see that this norm is a real number and if we define this norm to unit $Det\hat{Q} = 1$, it will define a cubic surface in $D = 9$.

5 Real ternary Tu3-algebra and root system

Let us give the link the nonions with the canonical $SU(3)$ matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{3}(q_1 + q_2 + q_3 + q_4 + j q_5 + q_6) \tag{119}$$

$$\lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{i}{3}(-q_1 - q_2 - q_3 + q_4 + j q_5 + j^2 q_6) \tag{120}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \frac{1}{3}(q_1 + j^2 q_2 + j q_3 + q_4 + j q_5 + j^2 q_6) \tag{121}$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} = \frac{i}{3}(-q_1 - j^2 q_2 - j q_3 + q_4 + j q_5 + j^2 q_6) \tag{122}$$

$$\lambda_6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \frac{i}{3}(q_1 + j q_2 + j^2 q_3 + q_4 + j^2 q_5 + j q_6) \tag{123}$$

$$\lambda_7 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} = \frac{i}{3}(q_1 + j q_2 + j^2 q_3 - q_4 - j^2 q_5 - j q_6) \tag{124}$$

$$\lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{(1-j)}(q_7 - j q_8) \tag{125}$$

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} = \frac{-1}{\sqrt{3}}(j q_7 + j^2 q_8) \tag{126}$$

Here the matrices $\lambda_i/2 = g_i$ satisfy to ordinary $SU(3)$ algebra:

$$[g_i, g_j]_{Z_2} = i f_{ijk} g_k. \tag{127}$$

where f_{ijk} are completely antisymmetric and have the following values:

$$f_{123} = 1, f_{147} = f_{165} = f_{246} = f_{257} = f_{345} = f_{376} = \frac{1}{2}, f_{458} = f_{678} = \frac{\sqrt{3}}{2}. \tag{128}$$

Now we introduce the plus-step operators:

$$Q_1 = Q_I^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_2 = Q_{II}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_3 = Q_{III}^+ = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{129}$$

and on the minus-step operators:

$$Q_4 = Q_I^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_5 = Q_{II}^- = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad Q_6 = Q_{III}^- = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{130}$$

We choose the following 3-diagonal operators :

$$Q_7 = Q_I^0 = \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix} \qquad Q_8 = Q_{II}^0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad Q_0 = Q_{III}^0 = \begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}, \tag{131}$$

which produce the ternary Cartan subalgebra:

$$\{Q_0, Q_7, Q_8\} = 0. \tag{132}$$

The Q_k operators with $k = 0, 1, 2, ..., 8$ satisfy to the following ternary S_3 commutation relations:

N	$\{klm\} \to \{n\}$	f_{klm}^n	N	$\{klm\} \to \{n\}$	f_{klm}^n	N	$\{klm\} \to \{n\}$	f_{klm}^n
1	$\{0,1,2\} \to \{6\}$	$\frac{1}{\sqrt{3}}$	29	$\{1,2,3\} \to \{0\}$	$\sqrt{3}$	57	$\{2,4,7\} \to \emptyset$	0
2	$\{0,1,3\} \to \{5\}$	$-\frac{1}{\sqrt{3}}$	30	$\{1,2,4\} \to \{2\}$	1	58	$\{2,4,8\} \to \emptyset$	0
3	$\{0,1,4\} \to \{0,7,8\}$	$\left\{0, 0, \sqrt{\frac{2}{3}}\right\}$	31	$\{1,2,5\} \to \{1\}$	1	59	$\{2,5,6\} \to \{6\}$	-1
4	$\{0,1,5\} \to \emptyset$	0	32	$\{1,2,6\} \to \emptyset$	0	60	$\{2,5,7\} \to \{0,7,8\}$	$\left\{\frac{3}{\sqrt{2}}, -1, -\frac{2}{\sqrt{3}}\right\}$
5	$\{0,1,6\} \to \emptyset$	0	33	$\{1,2,7\} \to \{6\}$	$-\sqrt{\frac{2}{3}}$	61	$\{2,5,8\} \to \{0,7,8\}$	$\left\{-\sqrt{\frac{3}{2}}, 0, 1\right\}$
6	$\{0,1,7\} \to \emptyset$	0	34	$\{1,2,8\} \to \{6\}$	$\sqrt{2}$	62	$\{2,6,7\} \to \emptyset$	0
7	$\{0,1,8\} \to \{1\}$	$-\sqrt{\frac{2}{3}}$	35	$\{1,3,4\} \to \{3\}$	-1	63	$\{2,6,8\} \to \emptyset$	0
8	$\{0,2,3\} \to \{4\}$	$\frac{1}{\sqrt{3}}$	36	$\{1,3,5\} \to \emptyset$	0	64	$\{2,7,8\} \to \{2\}$	$\frac{1}{\sqrt{3}}$
9	$\{0,2,4\} \to \emptyset$	0	37	$\{1,3,6\} \to \{1\}$	-1	65	$\{3,4,5\} \to \emptyset$	0
10	$\{0,2,5\} \to \{0,7,8\}$	$\left\{0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\right\}$	38	$\{1,3,7\} \to \{5\}$	$\sqrt{\frac{2}{3}}$	66	$\{3,4,6\} \to \{4\}$	1
11	$\{0,2,6\} \to \emptyset$	0	39	$\{1,3,8\} \to \{5\}$	$\sqrt{2}$	67	$\{3,4,7\} \to \emptyset$	0
12	$\{0,2,7\} \to \{2\}$	$-\frac{1}{\sqrt{2}}$	40	$\{1,4,5\} \to \{5\}$	-1	68	$\{3,4,8\} \to \emptyset$	0
13	$\{0,2,8\} \to \{2\}$	$\frac{1}{\sqrt{6}}$	41	$\{1,4,6\} \to \{6\}$	1	69	$\{3,5,6\} \to \{5\}$	-1
14	$\{0,3,4\} \to \emptyset$	0	42	$\{1,4,7\} \to \{0,7,8\}$	$\left\{0, 0, \frac{1}{\sqrt{3}}\right\}$	70	$\{3,5,7\} \to \emptyset$	0
15	$\{0,3,5\} \to \emptyset$	0	43	$\{1,4,8\} \to \{0,7,8\}$	$\left\{\sqrt{6}, \sqrt{3}, 0\right\}$	71	$\{3,5,8\} \to \emptyset$	0
16	$\{0,3,6\} \to \{0,7,8\}$	$\left\{0, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{6}}\right\}$	44	$\{1,5,6\} \to \emptyset$	0	72	$\{3,6,7\} \to \{0,7,8\}$	$\left\{-\frac{3}{\sqrt{2}}, 1, -\frac{2}{\sqrt{3}}\right\}$
17	$\{0,3,7\} \to \{3\}$	$\frac{1}{\sqrt{2}}$	45	$\{1,5,7\} \to \emptyset$	0	73	$\{3,6,8\} \to \{0,7,8\}$	$\left\{-\sqrt{\frac{3}{2}}, 0, -1\right\}$
18	$\{0,3,8\} \to \{3\}$	$\frac{1}{\sqrt{6}}$	46	$\{1,5,8\} \to \emptyset$	0	74	$\{3,7,8\} \to \{3\}$	$\frac{1}{\sqrt{3}}$
19	$\{0,4,5\} \to \{3\}$	$-\frac{1}{\sqrt{3}}$	47	$\{1,6,7\} \to \emptyset$	0	75	$\{4,5,6\} \to \{0\}$	$-\sqrt{3}$
20	$\{0,4,6\} \to \{2\}$	$\frac{1}{\sqrt{3}}$	48	$\{1,6,8\} \to \emptyset$	0	76	$\{4,5,7\} \to \{3\}$	$\sqrt{\frac{2}{3}}$
21	$\{0,4,7\} \to \emptyset$	0	49	$\{1,7,8\} \to \{1\}$	$\frac{1}{\sqrt{3}}$	77	$\{4,5,8\} \to \{3\}$	$-\sqrt{2}$
22	$\{0,4,8\} \to \{4\}$	$\sqrt{\frac{2}{3}}$	50	$\{2,3,4\} \to \emptyset$	0	78	$\{4,6,7\} \to \{2\}$	$-\sqrt{\frac{2}{3}}$
23	$\{0,5,6\} \to \{1\}$	$-\frac{1}{\sqrt{3}}$	51	$\{2,3,5\} \to \{3\}$	1	79	$\{4,6,8\} \to \{2\}$	$-\sqrt{2}$
24	$\{0,5,7\} \to \{5\}$	$\frac{1}{\sqrt{2}}$	52	$\{2,3,6\} \to \{2\}$	1	80	$\{4,7,8\} \to \{4\}$	$-\frac{1}{\sqrt{3}}$
25	$\{0,5,8\} \to \{5\}$	$-\frac{1}{\sqrt{6}}$	53	$\{2,3,7\} \to \{4\}$	$2\sqrt{\frac{2}{3}}$	81	$\{5,6,7\} \to \{1\}$	$-2\sqrt{\frac{2}{3}}$
26	$\{0,6,7\} \to \{6\}$	$-\frac{1}{\sqrt{2}}$	54	$\{2,3,8\} \to \emptyset$	0	82	$\{5,6,8\} \to \emptyset$	0
27	$\{0,6,8\} \to \{6\}$	$-\frac{1}{\sqrt{6}}$	55	$\{2,4,5\} \to \{4\}$	-1	83	$\{5,7,8\} \to \{5\}$	$-\frac{1}{\sqrt{3}}$
28	$\{0,7,8\} \to \emptyset$	0	56	$\{2,4,6\} \to \emptyset$	0	84	$\{6,7,8\} \to \{6\}$	$-\frac{1}{\sqrt{3}}$

We have got 84 commutations relations. One commutation relation, $\{Q_0, Q_7, Q_8\}_{S_3} = 0$, provide the Cartan subalgebra. Let separate the rest 83 commutation relations on the 5 groups$(18+18+27+18+2)$. The first group contains itself the following 18 commutation relations in one group (see Table):

$$\begin{aligned} \{\vec{H}_\alpha, Q_1\}_{S_3} &= \vec{\alpha}_1 Q_1, \qquad & \{\vec{H}_\alpha, Q_4\}_{S_3} &= \vec{\alpha}_4 Q_4, \\ \{\vec{H}_\alpha, Q_2\}_{S_3} &= \vec{\alpha}_2 Q_2, \qquad & \{\vec{H}_\alpha, Q_5\}_{S_3} &= \vec{\alpha}_5 Q_5 \\ \{\vec{H}_\alpha, Q_3\}_{S_3} &= \vec{\alpha}_3 Q_3, \qquad & \{\vec{H}_\alpha, Q_6\}_{S_3} &= \vec{\alpha}_6 Q_6 \end{aligned} \tag{133}$$

Table 2: *I-The root system of the step operators*

$\{klm\} \to \{n\}$	f^n_{klm}	$\{klm\} \to \{n\}$	f^n_{klm}	$\{klm\} \to \{n\}$	f^n_{klm}	$\vec{\alpha}_i$
$\{1,7,8\} \to \{1\}$	$\frac{1}{\sqrt{3}}$	$\{2,7,8\} \to \{2\}$	$\frac{1}{\sqrt{3}}$	$\{3,7,8\} \to \{3\}$	$\frac{1}{\sqrt{3}}$	$\vec{\alpha}_1$
$\{4,7,8\} \to \{4\}$	$-\frac{1}{\sqrt{3}}$	$\{5,7,8\} \to \{5\}$	$-\frac{1}{\sqrt{3}}$	$\{6,7,8\} \to \{6\}$	$-\frac{1}{\sqrt{3}}$	$\vec{\alpha}_4$
$\{0,1,7\} \to \emptyset$	0	$\{0,2,7\} \to \{2\}$	$-\frac{1}{\sqrt{2}}$	$\{0,3,7\} \to \{3\}$	$\frac{1}{\sqrt{2}}$	$\vec{\alpha}_2$
$\{0,4,7\} \to \emptyset$	0	$\{0,5,7\} \to \{5\}$	$\frac{1}{\sqrt{2}}$	$\{0,6,7\} \to \{6\}$	$-\frac{1}{\sqrt{2}}$	$\vec{\alpha}_5$
$\{0,1,8\} \to \{1\}$	$-\sqrt{\frac{2}{3}}$	$\{0,2,8\} \to \{2\}$	$\frac{1}{\sqrt{6}}$	$\{0,3,8\} \to \{3\}$	$\frac{1}{\sqrt{6}}$	$\vec{\alpha}_3$
$\{0,4,8\} \to \{4\}$	$\sqrt{\frac{2}{3}}$	$\{0,5,8\} \to \{5\}$	$-\frac{1}{\sqrt{6}}$	$\{0,6,8\} \to \{6\}$	$-\frac{1}{\sqrt{6}}$	$\vec{\alpha}_6$

where $\vec{H}_\alpha = \{H_1, H_2, H_3\} = \{(Q_7, Q_8), (Q_0, Q_7), (Q_0, Q_8)\}$ and

$$\begin{aligned} \vec{\alpha}_1 &= -\vec{\alpha}_4 = \{\frac{1}{\sqrt{3}}, 0, -\sqrt{\frac{2}{3}},\} \\ \vec{\alpha}_2 &= -\vec{\alpha}_5 = \{\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\} \\ \vec{\alpha}_3 &= -\alpha_6 = \{\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}\}, \end{aligned} \tag{134}$$

where

$$\begin{aligned} &< \vec{\alpha}_i, \vec{\alpha}_i >= 1, \qquad i = 1, 2, ..., 6, \\ &< \vec{\alpha}_i, \vec{\alpha}_j >= 0, \qquad i, j = 1, 2, ..., 6, \qquad i \neq j. \end{aligned} \tag{135}$$

Note, that

$$\begin{aligned} &< \vec{\alpha}^0_i, \vec{\alpha}^0_i >= \frac{2}{3}, \qquad i = 1, 2, ..., 6, \\ &\vec{\alpha}^0_1 + \vec{\alpha}^0_2 + \vec{\alpha}^0_3 = 0, \\ &< \vec{\alpha}^0_1, \vec{\alpha}^0_2 >=< \vec{\alpha}^0_2, \vec{\alpha}^0_3 >=< \vec{\alpha}^0_3, \vec{\alpha}^0_1 >= -\frac{1}{3}, \end{aligned} \tag{136}$$

where

$$\vec{\alpha}^0_i = \{0, \vec{\alpha}_{i2}, \vec{\alpha}_{i3}\}, \qquad i = 1, 2, ..., 6, \tag{137}$$

are the binary non-zero roots, in which the first components $\vec{\alpha}_{i1}$, $i = 1, ..., 6$, are equal zero. Thus, we have got

$$\frac{< \vec{\alpha}_i, \vec{\alpha}_i >}{< \vec{\alpha}^0_i, \vec{\alpha}^0_i >} = \frac{3}{2} \tag{138}$$

Table 3: *II-The dual roots*

$\{klm\} \to \{n\}$	f^n_{klm}	$\{klm\} \to \{n\}$	f^n_{klm}	$\{klm\} \to \{n\}$	f^n_{klm}	$\vec{\beta}_i$
$\{0,1,2\} \to \{6\}$	$\frac{1}{\sqrt{3}}$	$\{7,1,2\} \to \{6\}$	$-\frac{\sqrt{2}}{\sqrt{3}}$	$\{8,1,2\} \to \{6\}$	$\sqrt{2}$	$\vec{\beta}_1$
$\{0,4,5\} \to \{3\}$	$-\frac{1}{\sqrt{3}}$	$\{7,4,5\} \to \{3\}$	$\frac{\sqrt{2}}{\sqrt{3}}$	$\{8,4,5\} \to \{3\}$	$-\sqrt{2}$	$\vec{\beta}_4$
$\{0,2,3\} \to \{4\}$	$\frac{1}{\sqrt{3}}$	$\{7,2,3\} \to \{4\}$	$\frac{2\sqrt{2}}{\sqrt{3}}$	$\{8,2,3\} \to \{\emptyset\}$	0	$\vec{\beta}_2$
$\{0,5,6\} \to \{1\}$	$-\frac{1}{\sqrt{3}}$	$\{7,5,6\} \to \{1\}$	$-\frac{2\sqrt{2}}{\sqrt{3}}$	$\{8,5,6\} \to \{\emptyset\}$	0	$\vec{\beta}_5$
$\{0,3,1\} \to \{5\}$	$\frac{1}{\sqrt{3}}$	$\{7,3,1\} \to \{5\}$	$-\frac{\sqrt{2}}{\sqrt{3}}$	$\{8,3,1\} \to \{5\}$	$-\sqrt{2}$	$\vec{\beta}_3$
$\{0,6,4\} \to \{2\}$	$-\frac{1}{\sqrt{3}}$	$\{7,6,4\} \to \{2\}$	$\frac{\sqrt{2}}{\sqrt{3}}$	$\{8,6,4\} \to \{2\}$	$\sqrt{2}$	$\vec{\beta}_6$

Note, that there is only one simple root, since all α_i, $i = 1,2,3$ or $i = 4,5,6$, are related by usual Z_2 transformations, $\vec{\alpha}_i = -\vec{\alpha}_{i+3}$, $(i = 1,2,3)$, or by Z_3 transformations:

$$\vec{\alpha}_2 = R^V(q)\vec{\alpha}_1 = O(2\pi/3)\vec{\alpha}_1, \qquad \alpha_3 = R^V(q)^2\vec{\alpha}_1 = O(4\pi/3)\vec{\alpha}_1, \tag{139}$$

where

$$R^V(q) = O(2\pi/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{140}$$

$R^V(q^2) = (R^V(q))^2$ and $(R^V(q))^3 = R^V(q_0)$

Now one can unify in second group the other 18 commutations relations:

Using the properties of multiplications:

$$\begin{array}{c} Q_6 = Q_1Q_2, Q_4 = Q_2Q_3, Q_5 = Q_3Q_1, \\ Q_3 = Q_4Q_5, Q_1 = Q_5Q_6, Q_2 = Q_6Q_4, \end{array} \tag{141}$$

one can introduce the new systems of the *beta*-roots (see Table II):

$$\begin{array}{l} \vec{\beta}_1 = -\vec{\beta}_4 = \{\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, \sqrt{2}\} \\ \vec{\beta}_2 = -\vec{\beta}_5 = \{\frac{1}{\sqrt{3}}, \frac{2\sqrt{2}}{\sqrt{3}}, 0\} \\ \vec{\beta}_3 = -\vec{\beta}_6 = \{\frac{1}{\sqrt{3}}, -\frac{\sqrt{2}}{\sqrt{3}}, -\sqrt{2}\} \end{array} \tag{142}$$

$$\begin{array}{ll} <\vec{\beta}_i, \vec{\beta}_i>= 3, & i = 1,2,...,6, \\ <\vec{\beta}_i, \vec{\beta}_j>= -1, & i,j = 1,2,3, \qquad i \neq j. \end{array} \tag{143}$$

Table 4: *III-The root system of the step operators*

$\{klm\} \to \{n\}$	f^n_{klm}	$\{klm\} \to \{n\}$	f^n_{klm}	$\{klm\} \to \{n\}$	f^n_{klm}
$\{1,4,0\} \to \{0,7,8\}$	$\{0,0,\frac{\sqrt{2}}{\sqrt{3}}\}$	$\{1,5,0\} \to \emptyset$	$\{0,0,0\}$	$\{1,6,0\} \to \emptyset$	$\{0,0,0\}$
$\{1,4,7\} \to \{0,7,8\}$	$\{0,0,\frac{1}{\sqrt{3}}\}$	$\{1,5,7\} \to \emptyset$	$\{0,0,0\}$	$\{1,6,7\} \to \emptyset$	$\{0,0,0\}$
$\{1,4,8\} \to \{0,7,8\}$	$\{\sqrt{6},\sqrt{3},0\}$	$\{1,5,8\} \to \emptyset$	$\{0,0,0\}$	$\{1,6,8\} \to \emptyset$	$\{0,0,0\}$
$\{2,4,0\} \to \emptyset$	$\{0,0,0\}$	$\{2,5,0\} \to \{0,7,8\}$	$\{0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{6}}\}$	$\{2,6,0\} \to \emptyset$	$\{0,0,0\}$
$\{2,4,7\} \to \emptyset$	$\{0,0,0\}$	$\{2,5,7\} \to \{0,7,8\}$	$\{\frac{3}{\sqrt{2}},-1,-\frac{2}{\sqrt{3}}\}$	$\{2,6,7\} \to \emptyset$	$\{0,0,0\}$
$\{2,4,8\} \to \emptyset$	$\{0,0,0\}$	$\{2,5,8\} \to \{0,7,8\}$	$\{-\frac{\sqrt{3}}{\sqrt{2}},0,1\}$	$\{2,6,8\} \to \emptyset$	$\{0,0,0\}$
$\{3,4,0\} \to \emptyset$	$\{0,0,0\}$	$\{3,5,0\} \to \emptyset$	$\{0,0,0\}$	$\{3,6,0\} \to \{0,7,8\}$	$\{0,-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{6}}\}$
$\{3,4,7\} \to \emptyset$	$\{0,0,0\}$	$\{3,5,7\} \to \emptyset$	$\{0,0,0\}$	$\{3,6,7\} \to \{0,7,8\}$	$\{-\frac{3}{\sqrt{2}},1,-\frac{2}{\sqrt{3}}\}$
$\{3,4,8\} \to \emptyset$	$\{0,0,0\}$	$\{3,5,8\} \to \emptyset$	$\{0,0,0\}$	$\{3,6,8\} \to \{0,7,8\}$	$\left\{-\sqrt{\frac{3}{2}},0,-1\right\}$

Table 5: *IV-The root system of the step operators*

$\{klm\} \to \{n\}$	f^n_{klm}	$\{klm\} \to \{n\}$	f^n_{klm}	$\{klm\} \to \{n\}$	f^n_{klm}
$\{1,2,4\} \to \{2\}$	$\{1\}$	$\{1,2,5\} \to \{1\}$	$\{1\}$	$\{1,2,6\} \to \{\emptyset\}$	$\{0\}$
$\{1,3,4\} \to \{3\}$	$\{-1\}$	$\{1,3,5\} \to \{\emptyset\}$	$\{0\}$	$\{1,3,6\} \to \{1\}$	$\{-1\}$
$\{2,3,4\} \to \{\emptyset\}$	$\{0\}$	$\{2,3,5\} \to \{3\}$	$\{1\}$	$\{2,3,6\} \to \{2\}$	$\{1\}$
$\{1,4,5\} \to \{5\}$	$\{-1\}$	$\{2,4,5\} \to \{4\}$	$\{-1\}$	$\{3,4,5\} \to \{\emptyset\}$	$\{0\}$
$\{1,4,6\} \to \{6\}$	$\{1\}$	$\{2,4,6\} \to \{\emptyset\}$	$\{0\}$	$\{3,4,6\} \to \{4\}$	$\{1\}$
$\{1,5,6\} \to \{\emptyset\}$	$\{0\}$	$\{2,5,6\} \to \{6\}$	$\{-1\}$	$\{3,5,6\} \to \{5\}$	$\{-1\}$

Note, that there is also only one simple dual root, since all β_i, $i = 1,2,3$ or $i = 4,5,6$, are related by usual Z_2 transformations, $\vec{\beta}_i = -\vec{\beta}_{i+3}$, $(i = 1,2,3)$, or by Z_3 transformations:

$$\vec{\beta}_2 = R^V(q)\vec{\beta}_1 = O(2\pi/3)\vec{\beta}_1, \qquad \beta_3 = R^V(q)^2\vec{\beta}_1 = O(4\pi/3)\vec{\beta}_1, \tag{144}$$

where

$$R^V(q) = O(2\pi/3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1/2 & \sqrt{3}/2 \\ 0 & -\sqrt{3}/2 & -1/2 \end{pmatrix}, \tag{145}$$

$R^V(q^2) = (R^V(q))^2$ and $(R^V(q))^3 = R^V(q_0)$

The third group contains itself 27 commutation relations amonth them there are only 9 have the non-zero results:

The fourth group has also the 18 commutations relations:

The last, 5-th , group has only two but very important commutation relations:

Table 6: *V-The root system of the step operators*

$\{klm\} \to \{n\}$	f^n_{klm}	$\{klm\} \to \{n\}$	f^n_{klm}
$\{1,2,3\} \to \{0\}$	$\{\sqrt{3}\}$	$\{4,5,6\} \to \{0\}$	$\{-\sqrt{3}\}$

6 C_N- Clifford algebra

We begin with a V, a finite-dimensional vector space over the fields, $\Lambda = R, C$ or $\Lambda = TC$.

We introduce the tensor algebra $T(V) = \oplus_{n\geq 0} \otimes^n V$, with $\otimes^0 V = \Lambda$.

The product in $T(V)$ one can define as follows: $v_1\otimes ...\otimes v_p \in V^{\otimes p}$ and $u_1 \otimes ... \otimes u_q \in V^{\otimes q}$, then their product is $v_1 \otimes ... \otimes v_p \otimes u_1 \otimes ... \otimes u_q \in V^{\otimes(p+q)}$. For example, if V has a basis $\{x, y\}$, then $T(V)$ has a basis $\{1, x, y, xy, yx, x^2, y^2, x^2y, y^2x, x^2y^2, ...\}$. Suppose now we introduce into V a trilinear form $(..., ..., ...)$. Let $J =< v \otimes v \otimes v - (v, v, v) \cdot 1 | v \in V >$ an ideal in $T(V)$ and put

$$TCl(V) = T(V)/J, \tag{146}$$

the Clifford algebra over V with trilinear form $(..., ..., ...)$.

To generalize the binary Clifford algebra one can introduce the following generators $q_1, q_2, ..., q_n$ and relations:

$$q_k^3 = 1 \tag{147}$$

and

$$q_k q_l = j q_l q_k, \qquad q_l q_k = j^2 q_k q_l, \qquad n \geq l > k \geq 1, \tag{148}$$

where $j = \exp(2\pi/3)$. One can immeadiately find two types of the S_3 identities. The first type of such identities are:

$$\begin{aligned} q_k q_l q_k + q_k^2 q_l + q_l q_k^2 &= (j + 1 + j^2) q_k^2 q_l = 0, \\ q_k q_l q_k + j^2 q_k^2 q_l + j q_l q_k^2 &= (j + j^2 + 1) q_k^2 q_l = 0, \\ q_k q_l q_k + j q_k^2 q_l + j^2 q_l q_k^2 &= (3j) q_k^2 q_l, \end{aligned} \tag{149}$$

or

$$\begin{aligned} q_l q_k q_l + q_1^2 q_k + q_k q_l^2 &= (j^2 + j + 1) q_k q_l^2 = 0, \\ q_l q_k q_l + j^2 q_1^2 q_k + j q_k q_l^2 &= (j^2 + 1 + j) q_k q_l^2 = 0, \\ q_l q_k q_l + j q_1^2 q_k + j^2 q_k q_l^2 &= (3j^2) q_k q_l^2, \end{aligned} \tag{150}$$

The second type of the identities relate to the triple product of the generators with all different indexes, for example , one can take take $n \geq m > l > k \geq 1$. Then one can easily get :

$$\begin{aligned} & (q_k q_l q_m + q_l q_m q_k + q_m q_k q_l) + (q_m q_l q_k + q_l q_k q_m + q_k q_m q_l) \\ = & \Big((1 + j^2 + j^2) + (1 + j + j) \Big) q_k q_l q_m = \Big((j^2 - j) + (j - j^2) \Big) q_k q_l q_m \\ = & \Big((1 + j + j) + (1 + j^2 + j^2) \Big) q_m q_l q_k = \Big((j - j^2) + (j^2 - j) \Big) q_m q_l q_k \\ = & \ 0. \end{aligned} \tag{151}$$

From these two types of the identities one can see, that S_{3+}-symmetric sum

$$\sum_{S_{3+}} = q_k q_l q_m + q_l q_m q_k + q_m q_k q_l + q_m q_l q_k + q_l q_k q_m + q_k q_m q_l = \{q_k q_l q_m\} \tag{152}$$

is not equal zero just in one case, when all indexes, k, l, m are equal, *i.e.*:

$$\sum_{S_{3+}} (q_k q_l q_m) = q_k q_l q_m + q_l q_m q_k + q_m q_k q_l + q_m q_l q_k + q_l q_k q_m + q_k q_m q_l = 6\delta_{klm}. \tag{153}$$

$$\sum_{k=1}^{k=n} \sum_{l=1}^{l=n} \sum_{m=1}^{m=n} (q_k q_l q_m) = \sum_{S_{3+}} (q_k q_l q_m) = (q_k q_l q_m + q_l q_m q_k + q_m q_k q_l + q_m q_l q_k + q_l q_k q_m + q_k q_m q_l) = n\delta_{klm}. \tag{154}$$

$TCl(V)$ is a Z_3-graded algebra. We put

$$T(V)_0 = \oplus_{n=3k} V^{\otimes n}, \qquad T(V)_1 = \oplus_{n=3k+1} V^{\otimes n}, \qquad T(V)_2 = \oplus_{n=3k+2} V^{\otimes n}. \tag{155}$$

Also, one can see

$$TCl(V)_0 = TCl(V)_0 \oplus TCl(V)_1 \oplus TCl(V)_2, \qquad TCl(V)_k = T(V)_k / J_k. \tag{156}$$

$$J_k = J \bigcap T(V)_k. \tag{157}$$

If $dim_\Lambda V = n$ and $\{q_1, ..., q_n\}$ is an ortghonal basis for V with $(q_k, q_l, q_m) = \lambda_k \delta_{k,l,m}$, then the dimension $dim_\Lambda TCl(V) = 3^n$ and $\{\prod q_k^{l_k}\}$ is a basis where l_k is 0,1,or 2.

$n-gen$	1	2	3	4	5	6
TCl_0	1	$1+2$	$1+7+1$	$1+16+10$	$1+30+45+5$	$1+50+141+50+1$
TCl_1	1	$2+1$	$3+6$	$4+19+4$	$5+45+30+1$	$-$
TCl_2	1	3	$6+3$	$1+16+1$	$15+51+15$	$-$
Σ	3	$3\times 3=9$	9×3	81	$81\times 3=243$	$243\times 3=729$

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0.	1	1
1.	$< q_1, q_2, q_3, q_4 >$	4
2.	$< q_1^2, q_2^2, q_3^2, q_4^2 >$	$-$
	$< q_1q_2, q_1q_3, q_1q_4, q_2q_3, q_2q_4, q_3q_4 >$	10
3.	$< q_1^2q_2, q_1^2q_3, q_1^2q_4, q_2^2q_1, q_2^2q_3, q_2^2q_4,$	$-$
	$q_3^2q_1, q_3^2q_2, q_3^2q_4, q_4^2q_1, q_4^2q_2, q_4^2q_3 >$	$-$
	$< q_1q_2q_3, q_1q_2q_4, q_1q_3q_4, q_2q_3q_4 >$	16
4.	$< q_1^2q_2^2, q_1^2q_3^2, q_1^2q_4^2, q_2^2q_3^2, q_2^2q_4^2, q_3^2q_4^2 >$	$-$
	$< q_1^2q_2q_3, q_1^2q_2q_4, q_1^2q_2q_3, q_2^2q_1q_3, q_2^2q_1q_4, q_2^2q_3q_4,$	$-$
	$q_2^2q_1q_3, q_2^2q_1q_4, q_2^2q_3q_4, q_2^2q_1q_3, q_2^2q_1q_4, q_2^2q_3q_4$	$-$
	$q_3^2q_1q_2, q_3^2q_1q_4, q_3^2q_2q_4, q_4^2q_1q_2, q_4^2q_1q_3, q_4^2q_2q_3 >$	$-$
	$< q_1q_2q_3q_4 >$	19
5.	$< q_1^2q_2^2q_3, q_1^2q_2^2q_4, q_1^2q_3^2q_2, q_1^2q_3^2q_4, q_1^2q_4^2q_2, q_1^2q_4^2q_3,$	$-$
	$q_2^2q_3^2q_1, q_2^2q_3^2q_4, q_2^2q_4^2q_1, q_2^2q_4^2q_3, q_3^2q_4^2q_1, q_3^2q_4^2q_2 >$	$-$
	$< q_1^2q_2q_3q_4, q_2^2q_1q_3q_4, q_3^2q_1q_2q_4, q_4^2q_1q_2q_3 >$	16
6.	$< q_1^2q_2^2q_3^2, q_1^2q_2^2q_4^2, q_1^2q_3^2q_4^2, q_2^2q_3^2q_4^2 >$	$-$
	$< q_1^2q_2^2q_3q_4, q_1^2q_3^2q_2q_4, q_1^2q_4^2q_2q_3, q_2^2q_3^2q_1q_4, q_2^2q_4^2q_1q_3, q_3^2q_4^2q_1q_2 >$	10
7.	$< q_1^2q_2^2q_3^2q_4, q_2^2q_3^2q_4^2q_1, q_3^2q_4^2q_1^2q_2, q_4^2q_1^2q_2^2q_3 >$	4
8	$< q_1^2q_2^2q_3^2q_4^2 >$	1

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